Analytical Solution for Space Charge Limited Current Emission from a Sharp Tip using Variational Methods

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Abstract: *In this paper, we shall present an analytical solution for current density from a sharp tip in the tip-to-plate configuration, with an applied dc voltage bias, using the methods of variational calculus. We shall also show that when the radius of the tip tends to infinity, we get back the current density formula for the classical Child-Langmuir (CL) law for a flat surface. The local current density near the tip is found to be much higher than the current density in the classical CL configuration. The results seem to be in agreement with the recent numerical simulations of the current density near a sharp tip.*

Keywords: Space Charge Limited Emission (SCLE); sharp tip; Child-Langmuir (CL) law; prolate spheroidal coordinates; Variational Calculus (VC).

Introduction

The classical Child-Langmuir law represents the fundamental limit to the current density J_{CL} that can be drawn in the steady state from parallel plate electrodes separated by a distance D in one dimension but extending infinitely in two other dimensions. From Poisson's equations, the potential ϕ across the electrodes must satisfy [1-3]

$$J_{CL} = \epsilon_0 \sqrt{2e/m} \nabla^2 \phi \sqrt{\phi}, \tag{1}$$

where ϵ_0 is the permittivity of free space, e is the electron charge, and m is the electron mass. For a planar geometry, solving (1) yields [1-3]

$$J_{CL} = \frac{4}{9} \epsilon_0 \sqrt{\frac{2e}{m}} \frac{V^{3/2}}{D^2}. \tag{2}$$

Although vast research has been done on the space charge limited (SCL) current in the other 1D models such as cylindrical and spherical [2-3] geometries, no exact solution from first principles exists for tip-to-plate geometry [4]. Numerical simulations show that SCL current density near the tip varies as ~ $1/D$, as opposed to the classical $1/D^2$ in Child-Langmuir law [4].

Darr and Garner recently used variational calculus (VC) to derive closed-form solutions for axially symmetric cylindrical and spherical geometries [4]. In this paper, we shall use VC to derive analytical solutions for SCL emission from a sharp tip and explore relevant limits.

Mathematical Derivation

We consider a sharp tip of radius R and a scaling factor a that is approximated as a hyperboloid emitter and is expressed in prolate spheroidal coordinates (u, v, φ) [3], such that

$$\begin{aligned} x &= a \sinh(u) \sin(v) \cos(\varphi) \\ y &= a \sinh(u) \sin(v) \sin(\varphi) \\ z &= a \cosh(u) \cos(v) \end{aligned} \tag{3}$$

Since the system of prolate spheroidal coordinates is orthogonal [5], we can the variational methods to solve for SCLE from such an orthogonal system by first noting that in the space charge limit

$$\nabla^2 \phi = \frac{|\nabla \phi|^2}{4\phi} \tag{4}$$

We shall assume symmetry in the azimuthal direction such that all derivatives with respect to φ are zero. We then have, in prolate spheroidal coordinates,

$$|\nabla \phi|^2 = \frac{1}{a^2(\sinh^2(u) + \sin^2(v))} \left\{ \left(\frac{\partial \phi}{\partial u} \right)^2 + \left(\frac{\partial \phi}{\partial v} \right)^2 \right\} \tag{5}$$

The Laplacian in these coordinates is given by

$$\nabla^2 \phi = \frac{1}{a^2(\sinh^2(u) + \sin^2(v))} \left\{ \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} + \coth(u) \frac{\partial \phi}{\partial u} + \cot(v) \frac{\partial \phi}{\partial v} \right\} \tag{6}$$

Combining (3) – (6) yields

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} + \coth(u) \frac{\partial \phi}{\partial u} + \cot(v) \frac{\partial \phi}{\partial v} = \frac{1}{4\phi} \left\{ \left(\frac{\partial \phi}{\partial u} \right)^2 + \left(\frac{\partial \phi}{\partial v} \right)^2 \right\} \tag{7}$$

Using separation of variables, we can rewrite (7) as

$$\frac{1}{X} \frac{\partial^2 X}{\partial u^2} + \frac{\coth(u)}{X} \frac{\partial X}{\partial u} - \frac{1}{4X^2} \left(\frac{\partial X}{\partial u} \right)^2 = -\frac{1}{Y} \frac{\partial^2 Y}{\partial v^2} - \frac{\cot(v)}{Y} \frac{\partial Y}{\partial v} + \frac{1}{4Y^2} \left(\frac{\partial Y}{\partial v} \right)^2 = k \tag{8}$$

Equation (8) has an analytic solution only for $k = 0$.The general solution is

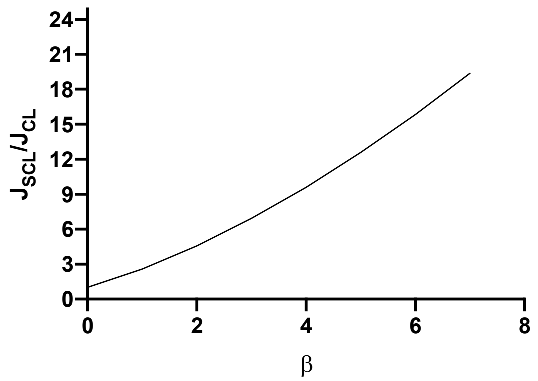

Figure 1. The graph of equation (13) with varying β

$$\phi(u,v) = c_0\left(c_1 + 3\ln\left(\tan\frac{v}{2}\right)\right)^{4/3}\left(c_2 + 3\ln\left(\tanh\frac{u}{2}\right)\right)^{4/3} \tag{9}$$

with c_1 and c_2 constants of integration that may be determined from the boundary conditions $\phi(u,v_c) = 0$ and $\phi(u,\pi/2) = V_g$, with V_g the gap voltage. Taking the limit of $\phi(u,\pi/2) = V_g$ as $u \to \infty$ with no edge effects gives

$$\phi(u,v) = V_g\left(1 - \frac{\ln\left(\tan\frac{v}{2}\right)}{\ln\left(\tan\frac{v_c}{2}\right)}\right)^{4/3} \tag{10}$$

To calculate the limiting current near the tip, we first evaluate $\nabla^2\phi\sqrt{\phi}$, at $u = 0$ and $v = v_c$ to obtain

$$J_{SCL}^{tip} = \epsilon_0\sqrt{2e/m}\,\frac{V_g^{\frac{3}{2}}}{a^2(\sin^2 v_c)}\,\frac{\csc^2\frac{v_c}{2}\sec^2\frac{v_c}{2}}{9\left(\ln\tan\frac{v_c}{2}\right)^2} \tag{11}$$

Defining $a^2 = D(D+R)$, we can simplify (11) and take the ratio to (2) to obtain

$$\frac{J_{SCL}^{tip}}{J_{CL}} = \frac{D(D+R)}{R^2\left(\ln\frac{\sqrt{R+D}+\sqrt{D}}{\sqrt{R}}\right)^2} \tag{12}$$

Defining $\cot^2 v_c = \sqrt{D/R} = \beta$ allows us to rewrite (12) as

$$\frac{J_{SCL}^{tip}}{J_{CL}} = \frac{\beta(1+\beta)}{\left(\ln\left(\sqrt{1+\beta}+\sqrt{\beta}\right)\right)^2} \tag{13}$$

Figure 1 shows J_{SCL}^{tip}/J_{CL}as a function of β. Taking the limit of (13) as $\beta \to 0^+$ yields $J_{SCL}^{tip}/J_{CL} \to 1$.
Table 1 shows J_{SCL} for R = 50 nm and V_g = 2500 V for various D using (13). Figure 2 shows ln J_{SCL} as a function of ln D, demonstrating that $J_{SCL} \propto 1/D^{0.55}$.

Table 1. Current density at the tip for as a function of *D* for *R* = 50 nm.

D **(nm)**	***J***

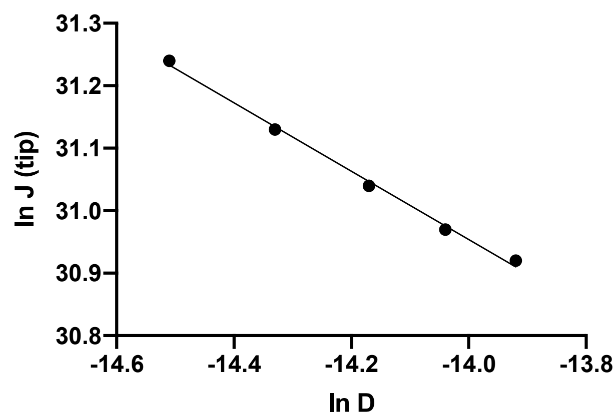

Figure 2. Space-charge limited current as a function gap distance showing that $J_{SCL} \propto D^{-0.55}$ using linear regression models from GraphPad's Prism 8.0.

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